JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 25, No. 2, March–April 2002

Turning Elliptic Orbital Planes via Intermediate Thrust Spherical Arcs

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Analytical solutions of the canonical equations for the variational problem of determining optimal spacecraft motion in a central Newtonian field are obtained. The solutions describe motion on a sphere with intermediate thrust and can satisfy necessary conditions for optimality. It is shown that such classes of intermediate thrust arcs can be used in the problem of minimizing the characteristic velocity of a maneuver to turn an elliptic orbit plane with respect to a given line having a known angle to the line of nodes. Computations show that, from the point of view of fuel expenditure, the turning maneuver via an intermediate thrust spherical arc is comparable to the cases of one, two, or three impulsive maneuvers.

Nomenclature

A_1, A_2	=	apogee locations
$a, \alpha, \lambda_0, \lambda_{10}$	=	constants
c	=	exhaust velocity
e, e'	=	eccentricities
f_H	=	true anomaly, deg
H, E	=	junctions where impulse is applied
i	=	inclination of elliptical orbit, deg
J	=	performance index
$\boldsymbol{l}$	=	unit thrust vector
LH	=	turning axes
l_1, l_2, l_3	=	components of unit thrust vector
m	=	mass of spacecraft, kg
P_1, P_2	=	perigee locations
p	=	semilatus rectum, km
Q_1, Q_2	=	location of nodes
q	=	dimensionless auxiliary control variable
$\boldsymbol{r}$	=	position vector
r_0	=	perigee range, km
r_1	=	apogee range, km
T	=	total maneuver time, s
t	=	flight time, s
u	=	angular distance from ascending node, deg
$\boldsymbol{v}$	=	velocity vector
$\boldsymbol{x}$	=	state vector
α_H	=	angle of turning, deg
β	=	mass flow rate, kg/s
$\triangle V$	=	characteristic velocity, km/s
$\triangle v$	=	magnitude of impulse, km/s
$\triangle\Lambda$	=	angle between lines Q_1Q_2 and $Q'_1Q'_2$, deg
δ	=	elevation, deg
θ	=	azimuth, deg
$\boldsymbol{\lambda}_r$	=	vector multiplier conjugated to spacecraft's position
$\boldsymbol{\lambda}_v$	=	primer vector
$\lambda_1, \lambda_2, \lambda_3$	=	components of primer vector
$\lambda_4, \lambda_5, \lambda_6$	=	multipliers associated with $\boldsymbol{\lambda}_r$
λ_7	=	multiplier conjugated to mass
μ	=	gravitational parameter of central body, km^3/s^2
ψ	=	angle between lines A_1P_1 and LH
Ω	=	longitude, deg
ω	=	argument of perigee, deg

Introduction

As is known, an optimal trajectory of a spacecraft moving with constant exhaust velocity and limited massflow rate in a Newtonian field may contain null thrust (NT), intermediate thrust (IT), and maximum thrust (MT) arcs.[1] The appearance of IT arcs is a degenerate case with added analytical difficulties connected with determining the state and control variables.[2,3] However, considerable progress has been made in the development of an analytical theory of IT arcs and their application to orbital transfer problems.

Early published works on IT arcs are due to Lawden.[1] In the case of free-flight time, Lawden obtained analytical solutions known as Lawden's spirals. It was later shown that IT arcs can be analyzed using two approaches: 1) a second variation test and 2) a transformation approach that provides a method with which to investigate members of a family of IT arcs, including Lawden's spiral.[2] Since the 1960s several works have been devoted to IT arcs. (See Refs. 3–5 for more details.)

Associated with the possibility of determining analytical solutions, there are other interesting questions about the optimality of IT arcs, their junction with other thrust arcs, and their applicability. Necessary conditions of optimality of IT arcs have been obtained using the magnitude of the primer vector,[4] the second variation test for the control parameters that lead to higher-order conditions, known as the generalized Legendre–Klebsch condition (Kelley–Contensou test) (see Refs. 2 and 5). On the basis of these solutions, it was shown that Lawden's spiral[2] is nonoptimal.[4,5] Sufficiency conditions of the optimality of IT arcs are associated with the existence conditions for a solution to the standard matrix Riccati equation (see Ref. 5).

Further investigations of optimal trajectories show that the equations of motion of a spacecraft and stationary conditions can be rewritten as a 14th-order canonical system of the equations of motion for each thrust arc.[6] Therefore, integration methods of analytical mechanics can be employed to find closed-form solutions for a thrust arc.[6,7] It is assumed that these solutions can describe a reference trajectory that further could be used in explicit guidance techniques. From this point of view, a revelation of principal difficulties connected with finding first integrals of the canonical system is especially important. It was shown that, in the case of three-dimensional IT arcs, the main difficulty of solving this system in quadratures is finding only one first integral.[6] The investigation of IT arcs through the canonical system of equations leads to a description of classes of analytical solutions. For example, analytic solutions for circular and spherical trajectories[8] and spiral trajectories[9,10] have been found.

One of the advantages of utilizing the canonical system is the application of Hamilton–Jacobi integration theory, which leads to

Received 27 July 2000; revision received 7 September 2001; accepted for publication 1 October 2001.

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analytical solutions for three-dimensional IT arcs through quadratures. It has been shown that the invariant relations of this system make it possible to find six classes of analytical solutions for spiral-shaped IT arcs.[11]

In regard to the optimality of IT arcs, note that extremals have been investigated in terms of satisfying the necessary conditions of optimality. In particular, it has been proven that some classes of spirals can satisfy these conditions. However, it has been shown that Lawden's spiral[2] is not a solution to the problem if the minimizing functional does not explicitly depend on angular distance.[9] Concerning the applicability of IT arcs, analytical solutions to fuel-optimal transfers between given intersecting and coaxial elliptical orbits using spiral IT arcs have been found.[9–11] The problem of spherical motions and their application for various types of boosting devices was discussed by Lurie[12] and it was shown that particular solutions for spherical trajectories can be described through the Levi–Civita method (see Ref. 8).

This paper is devoted to the investigation of spherical IT arcs, their optimality, and applicability. Analytic solutions for spherical IT arcs have been found using a Poisson bracket, invariant relations, and first integrals of the 14th-order canonical system of equations. These solutions generalize known solutions for the spherical motion with intermediate thrust. As an example, the problem of turning an elliptical orbit plane via a spherical IT arc is discussed. The functional to be minimized is the characteristic velocity of the spacecraft. It is known that the orbit transfer characteristics can be analyzed by the ideal impulsive thrust. This implies the velocity required to accomplish the maneuver is attained instantaneously without changing the spacecraft's position. This assumption may be admissible for a preliminary mission analysis but is not appropriate for analysis in the general guidance problem.[13] Thus, it is important to find explicit solutions for a thrust arc that would be used to achieve the mission objectives, instead of application of the impulsive thrust. In this context, the spherical IT arc is applied in the example to turn the elliptic orbit plane. For comparison, one-, two-, and three-impulse solutions are considered. It will be shown that for some turning angles and specified orbits, using the spherical IT arc is comparable with impulse turning.

Statement of the Problem

The equations of the Mayer variational problem of determining the spacecraft optimal motion in a central Newtonian field may be given in the form (see Refs. 1 and 7)

$$\dot{\boldsymbol{v}} = (c\beta/m)\boldsymbol{l} - (\mu/r^3)\boldsymbol{r}, \qquad \dot{\boldsymbol{r}} = \boldsymbol{v}, \qquad \dot{m} = -\beta, \qquad \dot{\boldsymbol{\lambda}} = -\boldsymbol{\lambda}_r$$
$$\dot{\boldsymbol{\lambda}}_r = (\mu/r^3)\boldsymbol{\lambda} - 3(\mu/r^5)(\boldsymbol{\lambda}^T\boldsymbol{r})\boldsymbol{r}, \qquad \dot{\lambda}_7 = (c\beta/m^2)\boldsymbol{\lambda}^T\boldsymbol{l} \quad (1)$$

where $\boldsymbol{r} = (r, 0, 0)$ is the radius vector of the spacecraft with the origin at the attracting center, $\boldsymbol{v} = (v_1, v_2, v_3)$ is the velocity vector, $\boldsymbol{l} = (l_1, l_2, l_3)$ is a unit thrust vector, m is the mass of the spacecraft, $\beta (0 \le \beta \le \bar{\beta})$ is the mass-flow rate, c is the constant exhaust velocity, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ is a vector, conjugated to $\boldsymbol{v}$ and known as the primer vector,[1] $\boldsymbol{\lambda}_r$ with the components defined in the next section denotes a vector multiplier conjugated to $\boldsymbol{r}$, and λ_7 is a multiplier conjugated to m. The components of all vectors are given in a spherical coordinate system (r, θ, δ) with the origin at the attracting center.

The convention adopted here is for vectors and matrices to be boldface and scalars to be italic type. For example, the magnitude of the vector $\boldsymbol{r}$ is denoted by r, and r_i represents the ith component of $\boldsymbol{r}$. The vector inner (dot) product is denoted by transpose, and the cross-product is denoted by $\times$.

The mass-flow rate β and the components of the unit thrust vector satisfy the equalities

$$\beta(\bar{\beta} - \beta) - q^2 = 0, \qquad l_1^2 + l_2^2 + l_3^2 = 1$$

We consider β, q, and l_i, $i = 1, 2, 3$, as piecewise continuous control functions. For simplification, all variables are denoted by x_i, $i = 1, \ldots, 7$, that is, the components of $\boldsymbol{v}$ are denoted by x_1, x_2, and x_3, the components of $\boldsymbol{r}$ are denoted by x_4, x_5, and x_6, and the rocket mass is denoted by x_7. It is assumed that the initial and final conditions are given by $x_j = x_{j0}$, $j = 1, \ldots, 7$, and $x_n = x_{n1}$, $n = 1, \ldots, l$, where $l < 7$.

It is required to find the time histories of β, q, and l_i such that x_i satisfies the equations of motion, the constraints for the control parameters indicated earlier, and the initial and final conditions and that minimizes the given functional $J(x_{l+1,1}, x_{l+2,1}, \ldots, x_{7,1})$.

From the analysis of the necessary conditions of optimality, it follows that, as already mentioned, the optimal trajectories may consist of NT, IT, and MT arcs.[1] It is also known from the analysis of the Weierstrass condition that[1]

$$\boldsymbol{l} = \boldsymbol{\lambda}/\lambda$$

from which it follows that the system in Eq. (1) can be rewritten in the following canonical form[2,6]:

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i}, \qquad \dot{\lambda}_i = -\frac{\partial H}{\partial x_i}, \qquad i = 1, \ldots, 7 \quad (2)$$

with the Hamiltonian

$$H = -(\mu/r^3)\boldsymbol{\lambda}^T\boldsymbol{r} + \boldsymbol{\lambda}_r^T\boldsymbol{v} + \chi\beta$$

where

$$\chi = (c/m)\boldsymbol{\lambda}^T\boldsymbol{l} - \lambda_7$$

is the switching function defined by the following conditions: 1) $\chi \le 0$ on the NT arcs, where $\beta = 0$; 2) $\chi = 0$ on the IT arcs, where $0 < \beta < \bar{\beta}$; and 3) $\chi \ge 0$ on the MT arcs, where $\beta = \bar{\beta}$.

If $q \ne 0$, the intermediate values of β are accessible. It has been shown that, if the thrust is constrained, the the optimal trajectory may not contain the IT arcs except for some particular cases.[4] An IT arc can be used as a part of the trajectory if impulsive thrust is allowed, whereas a combination of the NT and MT arcs is possible without need of impulsive thrust at junctions.[1,4] One can find some examples of extremal solutions that show that the combination of NT and IT arcs without impulsive thrust exist.[8–12] More information on the structure of the optimal trajectories may be found in Robbins's work.[4]

First Integrals and Invariant Relations

For IT arcs, the following first integrals hold[6]:

$$\lambda_1\left(-\frac{\mu}{r^2} + \frac{v_2^2}{r} + \frac{v_3^2}{r}\right) - \lambda_2\left(\frac{v_1 v_2}{r} - \frac{v_2 v_3}{r}\tan\delta\right)$$
$$- \lambda_3\left(\frac{v_2^2}{r}\tan\delta + \frac{v_1 v_3}{r}\right) + \lambda_4 v_1 + \lambda_5 \frac{v_2}{r\cos\delta} + \lambda_6\frac{v_3}{r} = C$$

$$\boldsymbol{v} \times \boldsymbol{\lambda} + \boldsymbol{r} \times \boldsymbol{\lambda}_r = \boldsymbol{A}(A_1, A_2, A_3) \quad (3)$$

or, expanding this equation, we have

$$\lambda_3 v_2(\cos\theta/\cos\delta) - \lambda_2 v_3(\cos\theta/\cos\delta)$$
$$- \lambda_5\cos\theta\tan\delta + \lambda_6\sin\theta = A_1 \quad (4)$$

$$\lambda_3 v_2(\sin\theta/\cos\delta) - \lambda_2 v_3(\sin\theta/\cos\delta)$$
$$- \lambda_5\sin\theta\tan\delta - \lambda_6\cos\theta = A_2 \quad (5)$$

$$\lambda_5 = A_3 \quad (6)$$

and also

$$\lambda_7 m = C_2 \quad (7)$$

$$\lambda^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = C_3 \quad (8)$$

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 - 2\lambda_4 r - c\lambda \ln(m_0/m) + 3Ct = C_1 \quad (9)$$

where

$$\boldsymbol{\lambda}_r = \left[\lambda_4, \lambda_1\frac{v_2}{r} + \frac{\lambda_2}{r}(v_3\tan\delta - v_1) - \lambda_3\frac{v_2}{r}\tan\delta\right.$$
$$\left. + \frac{\lambda_5}{r\cos\delta}, \lambda_1\frac{v_3}{r} - \lambda_3\frac{v_1}{r} + \frac{\lambda_6}{r}\right]$$

C, C_1, C_2, C_3, and $\boldsymbol{A}(A_1, A_2, A_3)$ are integration constants and λ_4, λ_5, and λ_6 are multipliers conjugated to r, θ, and δ, respectively. The integrals Eqs. (3–9) are well known in the literature in different forms.[1–8] In particular, the known complete set of the integrals has been described by Pines.[7] The integral in Eq. (3) is known as the Hamiltonian of the canonical system. The integrals Eqs. (4–6) are the scalar form of the vector integral that can be obtained directly from the spherical symmetry of the Newtonian field. The integral Eqs. (7–8) describe the main characteristics of the IT arcs, that is, the switching function is identically zero and the magnitude of the primer vector is constant over a nonzero time interval. The integral Eq. (9) is the only known integral that contains time explicitly and can be derived using Eqs. (1) and (3).

Use of the invariant relations is important in investigation of a canonical system.[14] For the arcs considered, the condition $\chi = 0$ must be satisfied in nonzero time intervals, that is, the derivatives of all orders of the switching function must vanish. In particular, the following equalities hold[4,9,12]:

$$\dot{\chi} = \boldsymbol{\lambda}^T \dot{\boldsymbol{\lambda}} = 0 \tag{10}$$

$$\ddot{\chi} = \dot{\boldsymbol{\lambda}}^T \dot{\boldsymbol{\lambda}} - \mu\left\{C_3^2 \big/ r^3 - 3[(\boldsymbol{\lambda}^T \boldsymbol{r})^2/r^5]\right\} = 0 \tag{11}$$

$$\chi^{(3)} = r^2\left[C_3^2 \boldsymbol{r}^T \dot{\boldsymbol{r}} + 2\boldsymbol{\lambda}^T \boldsymbol{r}(2\dot{\boldsymbol{\lambda}}^T \boldsymbol{r} + \boldsymbol{\lambda}^T \dot{\boldsymbol{r}})\right] - 5\boldsymbol{r}^T \dot{\boldsymbol{r}}(\boldsymbol{\lambda}^T \boldsymbol{r})^2 = 0 \tag{12}$$

$$\begin{aligned}\chi^{(4)} &= c\beta(\boldsymbol{\lambda}^T \boldsymbol{r})[3\lambda^2 r^2 - 5(\boldsymbol{\lambda}^T \boldsymbol{r})^2] + m\lambda[r^2\lambda^2 - 5(\boldsymbol{\lambda}^T \boldsymbol{r})^2] \\ &\quad \times (\dot{\boldsymbol{r}}^T \dot{\boldsymbol{r}} - \mu/r) + [3C(t - t_s) + C_3 c \ln(m_0/m) + C_1] r^2 \\ &\quad \times \left\{3C[(t - t_s) + C_3 c \ln(m_0/m)] + C_1 + \boldsymbol{\lambda}^T \dot{\boldsymbol{r}}\right\} - (\boldsymbol{\lambda}^T \boldsymbol{r})(\boldsymbol{r}^T \dot{\boldsymbol{r}}) \\ &\quad + (\boldsymbol{r}^T \dot{\boldsymbol{r}})[2\lambda^2 \boldsymbol{r}^T \dot{\boldsymbol{r}} - 5(\boldsymbol{\lambda}^T \boldsymbol{r})(\boldsymbol{\lambda}^T \dot{\boldsymbol{r}})] = 0\end{aligned} \tag{13}$$

Spherical IT Arcs and Their Optimality

It is known from Poisson's theorem that if $I_1 = I_1(x, \lambda, q_1)$ and $I_2 = I_2(x, \lambda, q_2)$ are two integrals of a canonical system, where q_1 and q_2 are arbitrary constants, then the Poisson bracket of these integrals is an integral of this system.[14] The theorem provides a new integral only if the I_1 and I_2 are specifically related to the canonical system. From the definitions, the invariant relations are constant during the motion. Thus, they can be used to form the Poisson bracket.[14] When the components of the vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}_r$ are taken into account, the relations Eqs. (10) and (11) may be rewritten in the form

$$I_1 = \boldsymbol{\lambda}^T \boldsymbol{\lambda}_r = 0, \qquad I_2 = \lambda_r^2 + 3(\mu/r^5)(\boldsymbol{\lambda}^T \boldsymbol{r})^2 - (\mu/r^3)\lambda^2 = 0$$

Forming Poisson's brackets from the left side of these equalities yields[14]

$$(I_1, I_2) = 9(\mu/r^5)(\boldsymbol{\lambda}^T \boldsymbol{r})\lambda^2 - 15(\mu/r^7)(\boldsymbol{\lambda}^T \boldsymbol{r})^3 = K_1 \tag{14}$$

and

$$\begin{aligned}[I_1, (I_1, I_2)] &= 105(\mu/r^9)(\boldsymbol{\lambda}^T \boldsymbol{r})^4 \\ &\quad - 90(\mu/r^7)(\boldsymbol{\lambda}^T \boldsymbol{r})^2\lambda^2 + 9(\mu/r^5)\lambda^4 = K_2\end{aligned} \tag{15}$$

where K_1 and K_2 are constants. The relation in Eq. (15) can be considered as a biquadratic equation in terms of $\boldsymbol{\lambda}^T \boldsymbol{r}$. The solution of this equation has the form

$$\boldsymbol{\lambda}^T \boldsymbol{r} = r\sqrt{\tfrac{3}{7}\lambda^2 + \sqrt{K_2 r^5 \big/ 105\mu + 24\lambda^4/245}} \tag{16}$$

Substituting Eq. (16) into Eq. (14) yields a 16th-degree equation in terms of r:

$$\begin{aligned}K_1^4 r^{16} &- \frac{15}{343} K_2^3 \mu r^{15} - \frac{18}{49} K_1^2 K_2 \mu \lambda^2 r^{13} - \frac{828}{343} K_2^2 \mu^2 \lambda^4 r^{10} + \frac{8460}{343} \\ &\times K_1^2 \mu^2 \lambda^6 r^8 - \frac{324}{2401} \mu^3 \lambda^8 \left(\frac{1409}{5} K_2 r^5 + \frac{1749091}{49} \mu \lambda^4\right) = 0\end{aligned}$$

It is known that if the first coefficient and last term of the polynomial equation with real coefficients have different signs, then the equation has at least one positive root. In this case, spherical motion will occur, that is, the relations in Eqs. (1–15) are satisfied by $r = \text{const}$.

Consider IT arcs that lie on a spherical layer with radius $r = \text{const}$. From Eq. (12), it follows that

$$(\boldsymbol{\lambda}^T \boldsymbol{r})(2\dot{\boldsymbol{\lambda}}^T \boldsymbol{r} + \boldsymbol{\lambda}^T \dot{\boldsymbol{r}}) = 0$$

which yields the equalities

$$\lambda_1 = \text{const}, \qquad -2\lambda_4 r + \lambda_2 v_2 + \lambda_3 v_3 = 0$$

$$\lambda_4 = 0, \qquad \lambda_2 v_2 + \lambda_3 v_3 = 0 \tag{17}$$

The second-order differential equation for the primer vector is given by[1]

$$\ddot{\boldsymbol{\lambda}} = -(\mu/r^3)\boldsymbol{\lambda} + (3\mu/r^5)(\boldsymbol{\lambda}^T \boldsymbol{r})\boldsymbol{r}$$

The projection of the equation for the primer vector in the direction of the radius vector yields the following relationships:

$$(c\beta/m)\lambda = 3(\mu/r^2)\lambda_1 \quad \text{if} \quad \lambda_1 > 0$$

$$-(c\beta/m)\lambda = 3(\mu/r^2)\lambda_1 \quad \text{if} \quad \lambda_1 < 0 \tag{18}$$

On the basis of Eq. (17), it follows that

$$C = (\mu/r^2)\lambda_1 \tag{19}$$

Here λ_1 and C must have the same signs. Integrating Eq. (18), and using $\chi = 0$, we obtain

$$m = m_0 e^{ht}, \qquad \lambda_7 = (c\lambda/m_0)e^{-ht} \tag{20}$$

where $\lambda_1 < 0$ and

$$h = 3\left(\mu\lambda_1 \big/ r^2 c\lambda\right)$$

The other solutions of the system Eq. (1) for IT arcs can be obtained in the following manner. First, differentiating the relation

$$\lambda_2^2 + \lambda_3^2 = \lambda^2 - \lambda_1^2 = \text{const} \tag{21}$$

and taking into account Eqs. (3) and (18), we obtain the equality

$$v_3 = \lambda_6 N_1 \tag{22}$$

where

$$N_1 = \frac{-3\lambda_1^2 + \lambda^2}{\lambda_1\left(3\lambda_1^2 + \lambda^2\right)}$$

Multiplying the relations of Eqs. (4) and (5) by $\sin\theta$ and $\cos\theta$, respectively, and subtracting the second relation from the first yields

$$\lambda_6 = a\sin(\theta - \alpha) \tag{23}$$

where a and α are integration constants related to A_1 and A_2 via

$$\tan\alpha = A_2/A_1, \qquad a = \sqrt{A_1^2 + A_2^2}$$

Substituting Eq. (1) into Eq. (23), and also taking into account

$$v_2^2 + v_3^2 = \mu/r - 3(\mu/r)\left(\lambda_1^2 \big/ \lambda^2\right) \tag{24}$$

which can be obtained with assistance of the equation for v_1 of the system Eq. (1) and relationship Eq. (19), leads to the relationships

$$v_3 = N_1 a \sin(\theta - \alpha), \qquad v_2 = \sqrt{N_2 - N_1^2 a^2 \sin^2(\theta - \alpha)} \tag{25}$$

where

$$N_2 = \mu\frac{-3\lambda_1^2 + \lambda^2}{r\lambda^2}$$

Then, using Eqs. (17) and (21), we find the components of the primer vector as

$$\lambda_2 = N_1 a\left(\sqrt{\lambda^2-\lambda_1^2}\Big/\sqrt{N_2}\right)\sin(\theta-\alpha)$$

$$\lambda_3 = \left(\sqrt{\lambda^2-\lambda_1^2}\Big/\sqrt{N_2}\right)\sqrt{N_2 - N_1^2 a^2 \sin^2(\theta-\alpha)} \quad (26)$$

To find a relationship between θ and δ, we differentiate Eq. (22) to obtain

$$\frac{v_2^2}{r}\tan\delta - \lambda_5\frac{N_1 v_2}{r\cos\delta}\tan\delta + \lambda_3\left[\frac{N_1\left(v_2^2+v_3^2\right)}{r\cos^2\delta} - \frac{c\beta}{m\lambda}\right] = 0$$

Solving for v_2 and comparing the result with the corresponding relation in Eq. (25) yields

$$N_2\tau + \lambda_5\frac{N_1\sqrt{N_2}}{\cos\delta}\tan\delta + \sqrt{\lambda^2-\lambda_1^2}\times\left[\frac{N_1\left(v_2^2+v_3^2\right)}{r\cos^2\delta} - \frac{rc\beta}{m\lambda}\right]\cot\delta = 0$$

When Eq. (25) is taken into account, θ and v_3 as a function of δ are found to be

$$\theta - \alpha = \arcsin\left[(N_2/a)\sqrt{(\mu/r)}N_3\right] \quad (27)$$

$$v_3 = \sqrt{\mu/r}\,N_3 \quad (28)$$

where

$$N_3 := \sqrt{\frac{s_1\sin^2 2\delta - (s_2 + s_3\cos^2\delta + s_4\sin\delta)^2}{s_5\sin^2 2\delta}}$$

$$s_1 := \frac{\lambda_1^2}{4}\left(-9\lambda_1^4+\lambda^4\right)^2, \qquad s_2 := \sqrt{\lambda^2-\lambda_1^2}\left(-9\lambda_1^4+\lambda^4\right)$$

$$s_3 := -3\lambda_1^2\sqrt{\lambda^2-\lambda_1^2}\left(3\lambda_1^2+\lambda^2\right)$$

$$s_4 := \lambda\lambda_5\frac{r}{\mu}\left(-3\lambda_1^2+\lambda^2\right)^{\frac{3}{2}}, \qquad s_5 := \lambda_1^2\lambda^2\frac{\left(-9\lambda_1^4+\lambda^4\right)}{4\left(-3\lambda_1^2+\lambda^2\right)} \quad (29)$$

Therefore, the variables v_2, v_3, λ_2, and λ_3 can also be expressed in terms of δ. Substituting Eq. (28) into the equation for the angle δ of system Eq. (1) yields

$$t - t_0 = \sqrt{\frac{r^3}{\mu}}\int\frac{\mathrm{d}\theta}{N_3} \quad (30)$$

where t_0 is an integration constant. For determination of boundary values for θ and δ, we can compute the derivative of Eq. (25) with respect to time and compare it with the third equation for $\dot{v}_3$ in system equation (1). Following this process leads to

$$N_1 a\cos(\theta-\alpha)(v_2/r\cos\delta) = \lambda_3(c\beta/m\lambda) - \left(v_2^2/r\right)\tan\delta$$

From this relation and using Eqs. (18), (25), and (26), at $\theta = 0$ we have

$$\sqrt{N_2}N_1 a\cos(\theta-\alpha) = 3\mu\sqrt{\lambda^2-\lambda_1^2}\Big/r^2\lambda^2$$

Consequently, as δ varies from $\delta_1 = 0$ to δ_2, the angle θ varies from

$$\theta_1 = \arcsin\sqrt{\frac{N_4}{N_5 N_1^2 a^2}} \qquad \text{to} \qquad \theta_2 = \arcsin\frac{N_2}{a}\sqrt{\frac{\mu}{r}}N_3$$

where

$$N_4 := 36\mu^2\lambda_1^2\left(\lambda^2-\lambda_1^2\right) - N_2^2 a^2 r^4\lambda^4$$

$$N_5 := 36\mu^2\lambda_1^2\left(\lambda^2-\lambda_1^2\right) - N_2 r^4\lambda^4$$

Let γ denote the angle between the velocity vector and the axis of the spherical system in the direction of increasing of θ, and let η denote the angle between a projection of the thrust vector on a tangent plane to the sphere and the axis of the spherical coordinate system in the direction of increasing of θ. These angles are given by

$$\tan\gamma = v_2/v_3, \qquad \tan\eta = \lambda_2/\lambda_3$$

From Eq. (25), it follows that $\gamma = \eta + \pi/2$. If $N_1^2 a^2 = N_2$, then

$$\gamma = \delta - \delta_0 + \pi/2, \qquad \eta = \delta - \delta_0 \quad (31)$$

It can be seen in Eq. (31) that for all IT arcs on a spherical layer, the direction of the thrust force always remains perpendicular to the direction of motion. Therefore, IT arcs on a spherical layer can be described by Eqs. (20), (23), (25–27), (30), and (31) in terms of the independent variable δ.

From Eq. (18), we see that λ_1 can be negative. This implies that the spherical trajectories obtained in this case can satisfy the necessary condition of optimality.[4] Also, from Eq. (18) it follows that nonoptimal spherical IT arcs exist, which confirms the known existence results.[2,4] Note that these solutions generalize the particular solutions obtained with using the Levi–Civita method (see Ref. 8). Indeed, for various values of the constants λ_1, a, and α in Eq. (23), one can obtain two sets of particular solutions. For instance, if the problem of minimization of the characteristic velocity is considered ($\lambda = 1$), for magnitudes $\lambda_1 = \frac{1}{3}$ and $a = 0$, from Eqs. (20), (23), (25), and (26), we obtain the following particular solutions:

$$v_1 = 0, \qquad v_2 = \sqrt{\frac{2}{3}\frac{\mu}{r}}, \qquad v_3 = 0, \qquad r = r_0$$

$$\theta = \sqrt{\frac{2\mu}{3r^3\cos^3\delta}}t + \theta_0, \qquad \delta = \delta_0, \qquad \lambda_2 = 0$$

$$\lambda_3 = \frac{2\sqrt{2}}{3}, \qquad \lambda_4 = 0, \qquad \lambda_5 = 0, \qquad \lambda_6 = 0$$

If $a = -\frac{2}{3}\sqrt{(2\mu)}r_0$ at $\lambda_1 = \frac{1}{3}$, then it follows that

$$v_1 = 0, \qquad v_2 = \sqrt{\frac{2\mu}{3r_0}[1 - 3\sin^2(\theta-\alpha)]}$$

$$v_3 = -\sqrt{\frac{2\mu}{r_0}}\sin(\theta-\alpha), \qquad \delta = \arcsin\left(\frac{1}{\sqrt{3}}\sin\nu\right)$$

$$\nu = \sqrt{\frac{6\mu}{r}}(t-t_0), \qquad \theta = \frac{\sqrt{1-3\sin^2\delta}}{\sqrt{3}\sin\delta} + \alpha$$

$$\lambda_2 = -\frac{2\sqrt{2}}{\sqrt{3}}\sin(\theta-\alpha)$$

$$\lambda_3 = -\frac{2\sqrt{2}}{\sqrt{3}}\sqrt{\frac{1}{3} - \sin^2(\theta-\alpha)}$$

$$\lambda_4 = 0, \qquad \lambda_5 = 0, \qquad \lambda_6 = -\frac{2}{3}\sqrt{\frac{2\mu}{r}}\sin(\theta-\alpha)$$

From the solutions, we also find that

$$m = m_0 e^{-h_0 t}, \qquad \lambda_7 = (c\lambda/m_0)e^{h_0 t}$$

where

$$h_0 = \mu\Big/r_0^2 c$$

Turning of Elliptical Orbit Plane via IT Arc

Consider the problem of turning an elliptical orbital plane in a central Newtonian field. It is known that this problem can be solved using one, two, and three impulses.[15] It has been shown that a maneuver using an IT arc obtained by the Levi–Civita method is coincident with one-impulse maneuvers and gives a considerable saving of fuel in comparison with a three-impulse manuever.[8] However, these IT

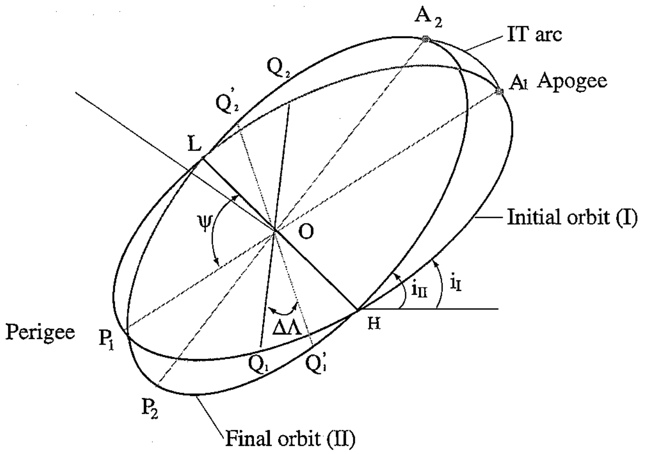

Fig. 1 Elliptic plane geometry.

arc solutions have been obtained for only certain values of eccentricity and components of the primer-vector. Realizing such transfers in more general cases is a problem of practical interest. Here it will be shown that the solutions for spherical IT arcs obtained in this paper can be applied to solving the problem of turning an elliptical orbit plane with arbitrary eccentricity. Components of the primer vector depend on the initial conditions.

Consider the problem of turning the plane of an elliptical orbit with an eccentricity e to angle α_H with respect to the LH line which is perpendicular to the line of nodes N_1N_2, as shown in Fig. 1. Denote the angle between the line of apsides AP of the initial orbit and the line LH by ψ. The location of any orbit with respect to the equitorial plane is determined by the angle of inclination i and longitude Ω measured from the vernal equinox. The initial and final conditions are given in the following form.

At $t = 0$:

$$v_1 = 0, \qquad v_2 = v_{21}, \qquad v_3 = v_{31}, \qquad r = r_0, \qquad \theta = \theta_1$$
$$\delta = \delta_1, \qquad m = m_0, \qquad i = i_1, \qquad \Omega = \Omega_1 \tag{32}$$

At $t = T$:

$$v_1 = 0, \qquad v_2 = v_{22}, \qquad v_3 = v_{32}, \qquad r = r_0, \qquad \theta = \theta_2$$
$$\delta = \delta_2, \qquad i = i_2, \qquad \Omega = \Omega_2 \tag{33}$$

As already noted, the maneuver of turning the plane can be realized by applying one, two, and three impulses. If we denote the initial orbit by I, the final orbit denote by II and if the impulse is applied at the point H of the orbit I, then it can be shown that the following formulas can be used to calculate the corresponding magnitudes of the impulses in each case.[15]

For one-impulse turning,

$$\triangle v_1 = 2\sqrt{[\mu/r_0(1-e)]}(1 - e\cos\psi)\sin(\alpha_H/2) \tag{34}$$

For two-impulse turning,

$$\triangle v_2 = \triangle v_{12} + \triangle v_{22}$$

where

$$\triangle v_{12} = 2v_{N_1}\sin(i_1/2), \qquad \triangle v_{22} = 2v_H \sin(i_2/2)$$

$$v_{N_1} = 2\sqrt{(\mu/p)}(1 + e\cos\omega)$$

$$v_H = \sqrt{(\mu/p)}[1 + e\cos(\omega + \triangle\omega)], \qquad \cot\triangle\omega = \sin i_2/\tan\alpha_H$$

For three-impulse turning,

$$\triangle v_3 = \triangle v_{13} + \triangle v_{23} + \triangle v_{33}$$

where

$$\triangle v_{13} = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2}{1 + r_1/r_0}} - \sqrt{1+e}\right), \qquad \triangle v_{23} = 2v_E \sin\frac{\alpha_H}{2}$$

$$\triangle v_{33} = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{1-e}{r_1/r_0}} - \frac{r_1}{r_0}\sqrt{\frac{2}{1 + r_1/r_0}}\right)$$

$$v_E = \sqrt{\frac{\mu}{r_1(1+e')}}(1 - e')\cos i_2, \qquad \frac{r_1}{r_0} = \frac{1-e'}{1+e'}$$

where r_0 is the radius of the sphere, r_1/r_0 is the ratio between the perigee range and apogee range, e is the eccentricity of the initial elliptical orbit, and e' is the eccentricity of the transfer orbit. The two- and three-impulse maneuvers contain one and two elliptical transfer orbits, respectively.[15] For example, the corresponding trajectory for the three-impulse maneuver consists of 1) the initial orbit, 2) a half-period first elliptical transfer orbit that lies in the plane of the initial orbit, 3) a half-period second elliptical transfer orbit that lies in the plane of the final orbit, and 4) the final orbit that starts at the perigee of the transfer orbit. The orbital transfer geometry of the maneuver via IT arc is simpler than these impulsive maneuvers and contains the initial and final orbits connected by the IT arc. More details on impulsive maneuvers and derivation of the formulas for the impulses may be found in the work of Ehricke.[15] These formulas will be used subsequently to compare dimensionless characteristic velocities of the impulsive maneuvers and the maneuver using IT arc. It will be assumed that the locations of the spacecraft on the initial and final orbits are not fixed, and therefore, the structure of the trajectory is defined to be NT–IT–NT.

It will be shown that the transfer from orbit I to orbit II can be realized via a spherical IT arc described by the solutions given in Eqs. (20), (23), (25–27), (30), and (31). According to Eq. (24), the velocity on the IT arc is constant and less than the local circular velocity and $v_1 = 0$. Consequently, motion on the IT arc will be started at the apogee A_1 of orbit I and completed at the point A_2 of orbit II. At point A_1, the thrust of the engine is applied and its magnitude, taking into account the law of changing mass Eq. (20), is determined by

$$F = \left(3\mu/r_0^2\lambda\right)m_0 e^{-ht}$$

The direction of the thrust force on the IT arc is determined utilizing Eq. (26). The spherical coordinates θ and δ of initial and final points, that is, points A_1 and A_2, of the IT arc depend on the terminal conditions Eqs. (32) and (33) and the turning angle α_H. If we denote an angular distance of the spacecraft from the ascending node in the orbital plane u, then the following relations are valid:

$$u_1 = \pi/2 + \psi, \qquad u_2 = u_H + \psi \tag{35}$$

For a spherical triangle, we have

$$\sin u_H = \sin i_1/\sin i_2 \tag{36}$$

If we consider that between the angles α_H, i_1, i_2, and $\triangle\Lambda$ the relationships

$$\sin i_1 = \tan\alpha_H \cot\triangle\Lambda, \qquad \sin\alpha_H = \sin i_2 \sin\triangle\Lambda \tag{37}$$

are valid, then using

$$\sin i_2 = \frac{\sin\alpha_H}{\sin[\arctan(\tan\alpha_H/\sin i_1)]} \tag{38}$$

we can rewrite the second relation of Eq. (35) taking into account Eq. (36) in the form

$$u_2 = \arcsin\frac{\sin i_1 \sin[\arctan(\tan\alpha_H/\sin i_1)]}{\sin\alpha_H} + \psi \tag{39}$$

Using Eq. (35) and Eqs. (37–39) and the relationships

$$\sin\delta = \sin i \sin u, \qquad \tan(\theta - \Omega) = \cos i \tan u \tag{40}$$

yields the formulas for determining the spherical coordinates of the initial and final points of the IT arc:

$$\theta_1 = \arctan(\cos i_1 \tan u_1) + \Omega_1$$

$$\theta_2 = \arctan(\cos i_2 \tan u_2) + \Omega_2$$

$$\delta_1 = \arcsin(\sin i_1 \cos \psi), \qquad \delta_2 = \arcsin(\sin i_2 \sin u_2) \quad (41)$$

where $\Omega_2 = \Omega_1 + \triangle\Lambda$ and the angle $\triangle\Lambda$, as follows from Eq. (37), is determined by

$$\tan \triangle\Lambda = \tan \alpha_H / \sin i_1 \quad (42)$$

The components of the velocity vector of the spacecraft at points A_1 and A_2 in the spherical coordinate system are given by

$$v_{21} = \sqrt{\frac{\mu}{r_0}(1-e)}\frac{\cos i_1}{\cos \delta_1}, \qquad v_{22} = \sqrt{\frac{\mu}{r_0}(1-e)}\frac{\cos i_2}{\cos \delta_2}$$

$$v_{21} = \sqrt{\frac{\mu}{r_0}(1-e)}\frac{\sin i_1 \cos u_1}{\cos \delta_1}$$

$$v_{21} = \sqrt{\frac{\mu}{r_0}(1-e)}\frac{\sin i_2 \cos u_2}{\cos \delta_2} \quad (43)$$

where the angles u_1, δ_1, u_2, δ_2, and i_2 are related to the terminal conditions and the angle α_H by Eqs. (35), (37), (38), and (40).

To conclude the development of the analytic solution, we determine the remaining constants that exist on the IT arc. Equating the right side of Eq. (24) to the velocity at the initial point of the IT arc [calculated using the first two formulas of Eq. (43)] yields

$$\lambda_{10} = \frac{\lambda_0^2}{3}[1-(1-e)]\left[\frac{\cos^2 i_1 + \sin^2 i_1 \cos^2 \psi}{\cos^2 \delta_1}\right] \quad (44)$$

Furthermore, equating the values of velocity v_3 at the initial and final points and taking into account Eqs. (25) and (43) leads to

$$N_1 a \sin(\theta_1 - \alpha) = \sqrt{\frac{\mu}{r_0}(1-e)}\frac{\sin i_1 \cos u_1}{\cos \delta_1}$$

$$N_1 a \sin(\theta_2 - \alpha) = \sqrt{\frac{\mu}{r_0}(1-e)}\frac{\sin i_2 \cos u_2}{\cos \delta_2} \quad (45)$$

from which it follows that

$$\tan \alpha = \frac{\sin \theta_1 - A \sin \theta_2}{\cos \theta_1 - A \cos \theta_2}$$

$$a = \sqrt{\frac{\mu}{r_0}(1-e)}\frac{\sin i_1 \cos u_1}{N_1 \sin(\theta_1 - \alpha)\cos \delta_1}$$

where

$$A = \frac{\sin i_1 \cos u_1 \cos \delta_2}{\sin i_2 \cos u_2 \cos \delta_1}$$

From Eqs. (27) and (45), we have

$$\lambda_5 = \frac{\mu}{r_0}\frac{\sqrt{s_1 \sin^2 2\delta_1 - s_7 \sin^2(\theta_2 - \alpha)\sin^2 2\delta_2} - (s_2 + s_3 \cos^2 \delta_2)}{s_6 \sin \delta_2}$$

where s_1, s_2, and s_3 are determined from Eq. (29) and

$$s_6 = \frac{\lambda_{10}^2\left(-3\lambda_{10}^2 + \lambda_0^2\right)s_5}{9\lambda_0^2\left(\lambda_0^2 - \lambda_{10}^2\right)}, \qquad s_7 = \left(-3\lambda_{10}^2 + \lambda_0^2\right)^{\frac{3}{2}}$$

The time of the maneuver start is computed by

$$T = \sqrt{r_0^3}\mu \int \sqrt{\frac{s_5 \sin^2 2\delta}{s_1 \sin^2 2\delta - \left[s_2 + s_3 \cos^2 \delta + \lambda\lambda_5\sqrt{r_0}\mu\left(-3\lambda_{10}^2 + \lambda_0^2\right)^{\frac{3}{2}} \sin\delta\right]^2}}\, d\delta$$

The characteristic velocity required for turning of the orbit plane can be determined by

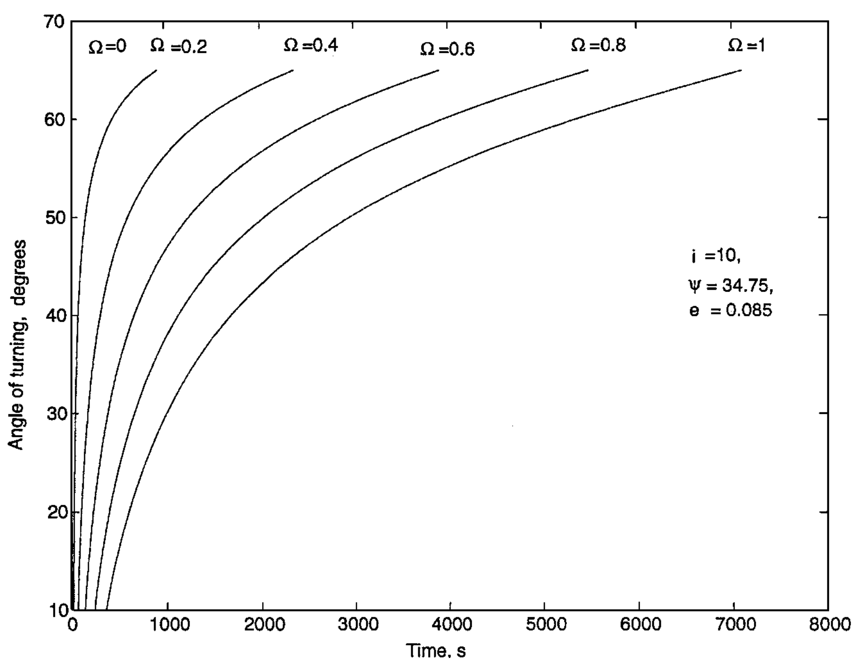

Fig. 2 Time vs turning angle for various values of longitude.

$$\triangle V = c\,\ln\left(\frac{m_0}{m_T}\right) = \frac{3\lambda_{10} v_H}{\lambda_0\sqrt{1-e}} \int \frac{s_1 \sin^2 2\delta - \left[s_2 + s_3\cos^2\delta + \lambda\lambda_5\sqrt{r_0/\mu}\left(-3\lambda_{10}^2 + \lambda_0^2\right)^{\frac{3}{2}}\sin\delta\right]^2}{s_5 \sin^2 2\delta}\,\mathrm{d}\delta \tag{46}$$

If the problem statement is to minimize the characteristic velocity, that is, if

$$\mathcal{J} = c\,\ln(m_0/m_T)$$

is considered, then from the transversality conditions, we have

$$\lambda_{71} = \frac{c}{m_T}, \qquad \lambda_5 = \frac{\partial J}{\partial \theta_1} = 0, \qquad \lambda_0 = 1$$

Thus, Eq. (46) can be integrated in terms of elementary functions resulting in

$$\triangle V = v_H[d_1(\theta_2 - \theta_1) + d_2(\cot 2\theta_1 - \cot 2\theta_2) + d_3(\cot\theta_1 - \cot\theta_2)]$$

where

$$d_1 = \frac{3\lambda_{10} v_H}{4 s_5 \lambda_0 \sqrt{1-e}}\left(s_1 + s_3^2\right), \qquad d_2 = -\frac{3\lambda_{10} v_H}{2 s_5 \lambda_0 \sqrt{1-e}} s_2^2$$

$$d_3 = \frac{3\lambda_{10} v_H}{4 s_5 \lambda_0 \sqrt{1-e}}\left(2 s_2 s_3 + s_3^2\right)$$

When these results and Eq. (20) are used, the flight time on the IT arc is

$$t_2 = \frac{r_0^2 \lambda_0 \triangle V}{3\mu\lambda_{10}}$$

Figure 2 shows the relationship between the flight time and α_H for various values of $\Omega\,(=\Omega_1)$ at fixed $i = (i_1)$, ψ, and e. As already indicated, the corresponding spherical IT arc satisfies the necessary conditions and end conditions and therefore may be considered as an extremal of the variational problem.

The ratio of the characteristic velocity required for one-, two- and three-impulse turning, and turning via the IT arc to the local circular velocity depends on values of parameters such as turning angle α_H, the angle of inclination i of the orbit to the main plane, the angle ψ between the lines LH and AP, and the longitude Ω, e, and λ_1. This ratio has been calculated for the following values:

$$r_0 = 10{,}000\,\text{km}, \qquad e_1 = 0.22, \qquad 0.08 \le e \le 0.1$$

$$0 \le \Omega \le 1.0, \qquad 5 \le i \le 25, \qquad 34.6 \le \psi \le 34.8$$

The computations for $e > 0.1$, $\alpha_H < 10$, $\alpha_H > 65$, $i > 25$, $\psi < 34.6$, and $\psi > 35$ are associated with significantly high values of the characteristic velocity, as shown in Fig. 3. Figure 4 shows the relationship between λ_1 and the initial orbit eccentricity, e. Note that, for comparison purposes, the formulas for computation of the magnitudes of the impulses for the one-, two-, and three-impulse maneuvers use the following values of parameters: $i = 10$, $\Omega = 0.6$, $\psi = 34.75$, and $e = 0.085$. It is assumed that if one of these parameters is varied, the other remaining three parameters remain fixed as indicated in Figs. 5–9. The results of the

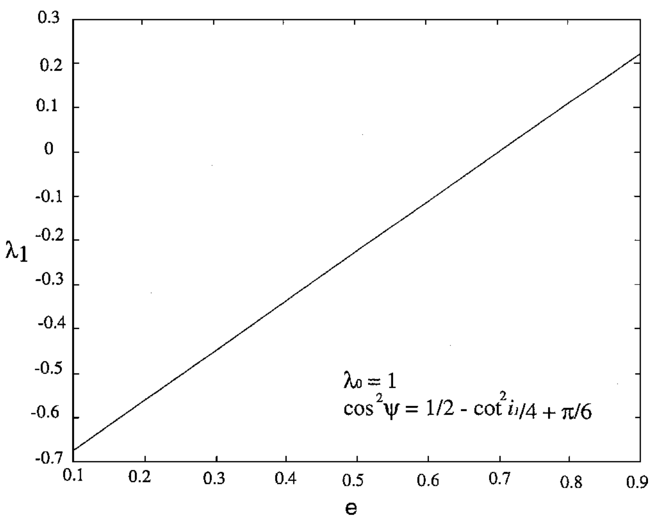

Fig. 4 Turning using an IT arc with λ_0 = 1.

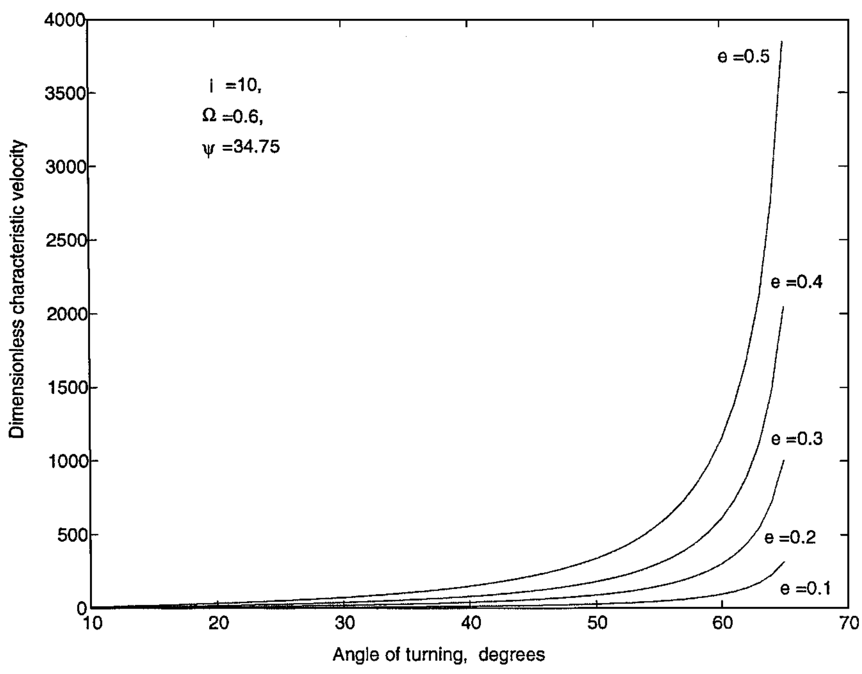

Fig. 3 Ratio $\triangle v_{\rm IT}/v_0$ vs turning angle α_H at higher eccentricities.

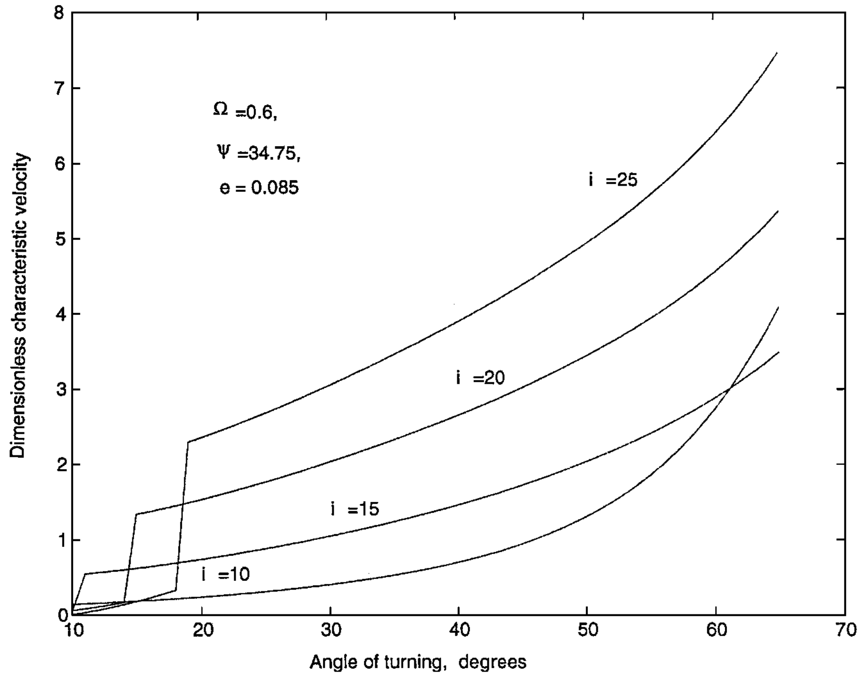

Fig. 5 Ratio $\triangle v_{IT}/v_0$ vs turning angle α_H for various inclinations.

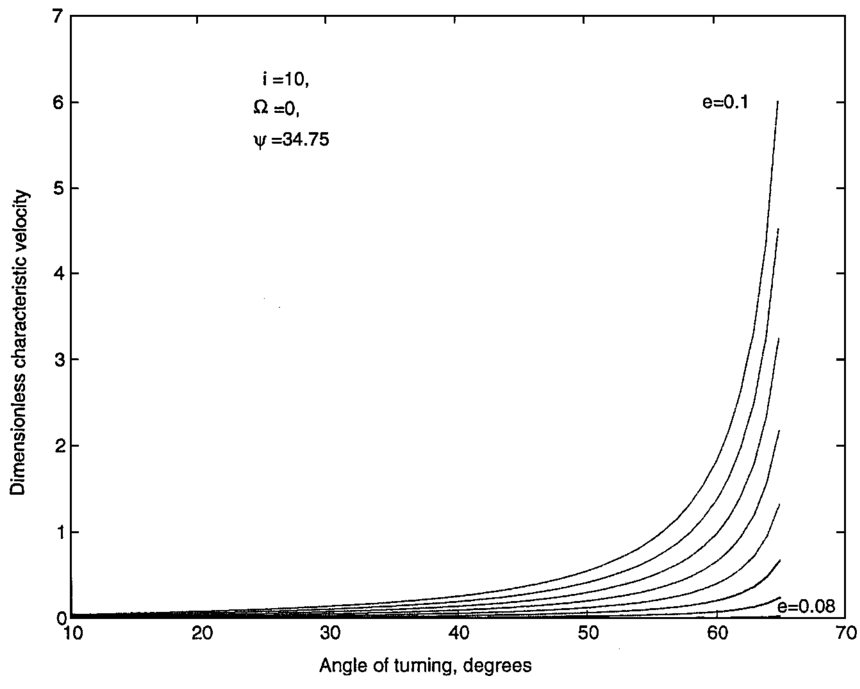

Fig. 6 Ratio $\triangle v_{IT}/v_0$ vs turning angle α_H for various eccentricities.

computations that reflect the main characteristics of the IT arcs and the dimensionless characteristic velocity are as follows:

1) Increasing angle i leads to an increase in the ratio $\triangle V_{IT}/v_0$. This dependency is shown in Fig. 5. Computations of the ratio for the one-, two-, and three-impulse maneuvers show that using the IT arc is more effective than the impulsive maneuvers if $10 \leq \alpha_H \leq 40$ and $0 \leq i \leq 15$ at the fixed Ω, ψ, and e given in Fig. 5.

2) Figure 6 shows that the ratio $\triangle V_{IT}/v_0$ increases as α_H increases and e decreases. Turning via an IT arc is more effective than impulsive maneuvers if $0 \leq e \leq 0.1$ and $10 \leq \alpha_H \leq 52$.

3) As mentioned earlier, the angle ψ changes within a small interval. As seen in Fig. 7, this behavior leads to a significantly decreasing ratio $\triangle V_{IT}/v_0$ that can be seen especially within $62 < \alpha_H < 65$. Computations show that a maneuver with an IT arc in this interval of ψ, while other parameters are fixed, is more effective than all three impulsive maneuvers.

4) Analysis of numerical results indicate that somewhat higher values of $\triangle V_{IT}/v_0$ may be obtained if Ω is varied within the interval $0 \leq \Omega \leq 1$ while α_H changes as the independent variable (see Fig. 8). In this case, comparisons of the dimensionless characteristic velocities show that maneuvers that use an IT arc are more effective than all three impulsive maneuvers if $0 \leq \Omega \leq 0.2$ at $10 \leq \alpha_H \leq 65$; $0 \leq \Omega \leq 0.4$ at $10 \leq \alpha_H \leq 55$; $0 \leq \Omega \leq 0.6$ at $10 \leq \alpha_H \leq 45$; $0 \leq \Omega \leq 0.8$ at $10 \leq \alpha_H \leq 35$; and $0 \leq \Omega \leq 1.0$ at $10 \leq \alpha_H \leq 25$. In particular, comparison of dimensionless characteristic velocities vs turning angles for the maneuvers using IT arc and one, two, and three impulses is shown in Fig. 9 for the values of all parameters indicated earlier.

5) The spherical arc with intermediate thrust is comparable when $22 \leq \alpha_H \leq 23$, $0.1 \leq e \leq 0.9$, $20 \leq \psi \leq 70$, and $\Omega = 0$, $10 \leq i \leq 80$. In these situations, the IT arc can replace the impulsive thrust.

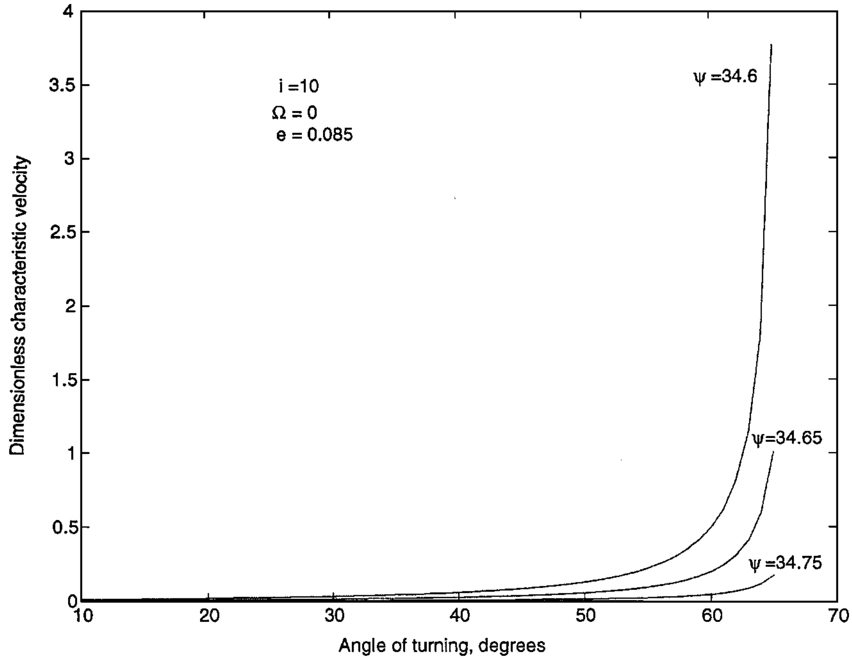

Fig. 7 Ratio $\triangle v_{\mathrm{IT}}/v_0$ vs turning angle α_H for various ψ.

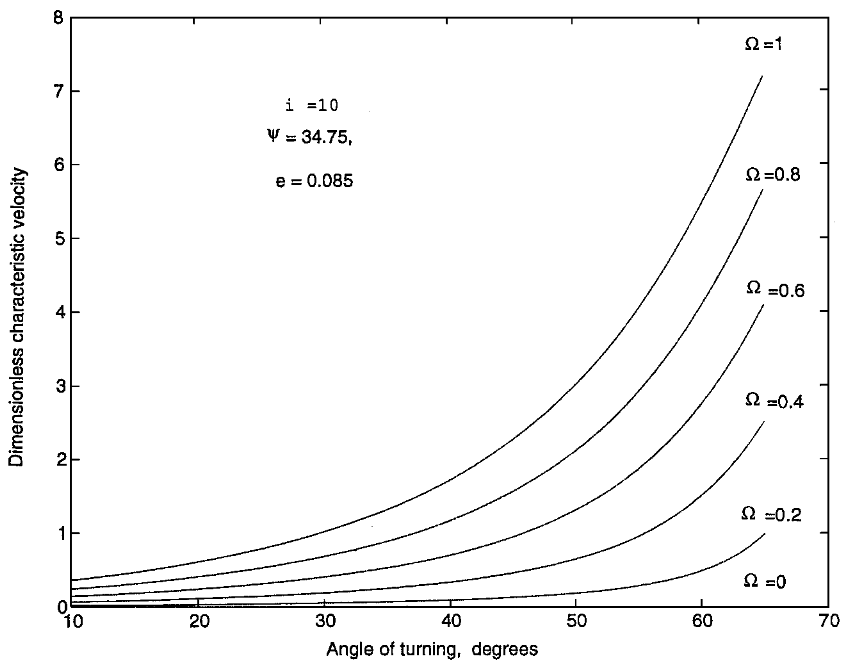

Fig. 8 Ratio $\triangle v_{\mathrm{IT}}/v_0$ vs turning angle α_H for various longitudes.

6) Using an intermediate thrust arc may be more effective than using a two-impulse maneuver for $\alpha_H \leq 25$, $0.1 \leq e_1 \leq 0.9$, $\Omega_1 = k(\pi/2)$, $k = 0, 1, 2, \ldots$, $10 \leq \psi \leq 90$, and $10 \leq i \leq 90$ and is comparable for $25 \leq \alpha_H \leq 30$.

7) Using an intermediate thrust arc may be more effective (or comparable with) a three-impulse manuever for $5 \leq \alpha_H < 60$, $0.1 < e < 0.3$, $10 \leq i < 90$, $10 \leq \psi \leq 40$, and $-360 \leq \Omega \leq 360$.

8) The other values of parameters can be used to compare the ratios of $\triangle v/v_0$. The corresponding computations show the following main characteristics in the behavior of characteristic velocities in each impulsive case, namely, $\triangle v_1$, $\triangle v_2$, and $\triangle v_3$. If i is increased then $\triangle v_1 = \text{const}$, $\triangle v_2$ is increased and $\triangle v_3$ is decreased. The functions $\triangle v_H$ and $\triangle v_3$ remain unchanged, and $\triangle v_2$ is changed by a small amount whereas Ω is increased within the interval $0 \leq \Omega \leq 1.0$. When e is changed within $0 \leq e \leq 0.1$, the function $\triangle v_1$ is decreased and $\triangle v_2$ and $\triangle v_3$ are increased.

Some other results of computations involving different values of parameters with different intervals are as follows.

9) There is almost a linear dependence between α_H and $\triangle V_{\mathrm{IT}}/v_0$ for $\Omega = 30$ and 120.

10) For any values of e, i, and ψ, there are discontinuities in values of t_2, $\triangle V_{\mathrm{IT}}/v_0$, m_2, λ_7, and $\theta_1 = k(\pi/2)$, $k = 0, 1, 2, \ldots$, at $10 \leq \psi \leq 60$ and $10 \leq i \leq 90$. In particular, for $\triangle V_{\mathrm{IT}}/v_0$, they can be seen in Fig. 5.

11) $\triangle V_{\mathrm{IT}}/v_0 < 0$ at $0 < \alpha_H < 22$ for any values of e, $\psi = 25.28$, and $\Omega = 0$.

12) As the numerical analysis shows, for the case when $i = 30$, $\psi = 30$, $\Omega = 0$, $e = 0.1$, $25 \leq \alpha_H \leq 85$, we have

$$0.0026 \leq \triangle V_{\mathrm{IT}}/v_0 \leq 0.02$$

with flight time $t_2 = 2.048$ s, which means that using the IT arc is more effective than using an impulsive maneuver

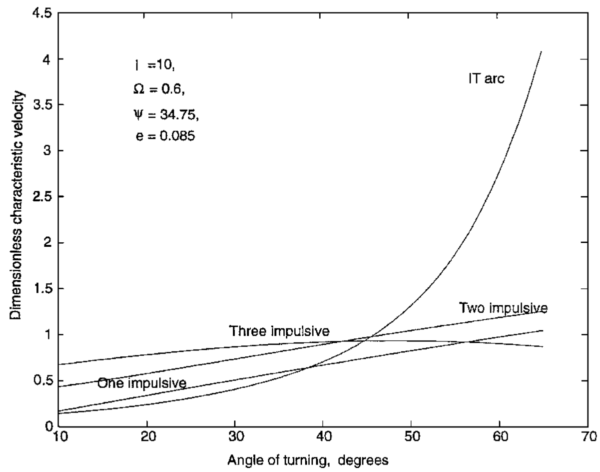

Fig. 9 Dimensionless characteristic velocities vs turning angle α_H for fixed parameters.

Conclusions

The variational problem on determination of optimal trajectories of a spacecraft in the central Newtonian field has been considered. Extremal analytical solutions for intermediate thrust arcs that lie on a spherical layer have been obtained. It was shown that the spherical IT arcs can be used to solve the problem of turning an elliptical orbit plane with arbitrary eccentricity. From the point of view of fuel expenditure, in certain terminal conditions a maneuver of turning an elliptic orbit plane via an IT spherical arc may be more effective than the cases of one-, two-, or three-impulse maneuvers.

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